

# Neuronal networks and controlled symmetries, a generic framework

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## Abstract

The extraordinary computational power of the brain may be related in part to the fact that each of the smaller neural networks that compose it can behave transiently in many different ways, depending on its inputs. We use contraction theory to extend earlier work on synchrony and symmetry, and exploit input continuity of contracting systems to ensure robust control of spatial and spatio-temporal symmetries of the output with the input.

## 1 Introduction

The brain is often described as being composed, in part, of a very large number of small “identical” functional units [1]. Cortical computation, for instance, is commonly viewed as being organized around cortical columns [2, 3, 4, 5]. In this context, a frequent suggestion is that the overall computational power of the brain may be related to some sort of combinatorial complexity [6], and to the fact that each part of the brain is reused for different computations.

At the level of individual units, although high behavioral variety could be explained by some learning process [7, 8, 9, 10, 11] or internal change, it is unlikely that such changes occur in very short periods of time. Instead, the most efficient source of behavioral variety could be simply to have a high input dependency, exploiting the nonlinearity of biological neural networks. In other words, depending on its input, a functional unit could behave in very diverse ways, though remaining stable and robust against noise and small variations in the input.

Intuitively, it is also known that sensory “input” enrolls only 5% or so of the connections to the thalamus [12], and that a similarly small percentage describes connections from the thalamus to input cortical layers [13]. Recent research suggests that this “paucity of input” between different regions of the brain is actually quite general [14].

Inspired by this, we will try to draw a generic framework allowing to observe neural systems under some “controlling” input. To this effect, we introduce “input continuity” analysis which will provide a way to describe the properties of a unit’s output knowing the properties of its input. This framework has broader possible uses and applications than used in this paper and will be discussed in the appendix. Here the main use of the “input continuity” will be its link with contracting systems [15], giving us a simple way to change the behavior of systems as will be shown with some toy examples in Section 5 also displaying some interesting results about symmetries and contracting systems, results exposed in Section 3 of the paper.

Indeed the study of symmetries is important in dynamical systems [16, 17] and more specifically in neural networks. It strongly influences, as we will see, synchrony and polysynchrony (concurrent synchronization), concepts playing an important role in neurobiology [18, 19, 20]. To this matter we will try to cover Lie continuous symmetries and spatio-temporal symmetries, giving some interesting tools to ensure this symmetries in the output. Most of those are based on contracting systems. Let us first recall different contraction theorem and properties.

## 2 Contraction

Essentially, a nonlinear time-varying dynamic system will be called *contracting* if initial conditions or temporary disturbances are forgotten exponentially fast, i.e., if trajectories of the perturbed system return to their nominal behavior with an exponential convergence rate. It turns out that relatively simple algebraic conditions can be given for this stability-like property to be verified, and that this property is preserved through basic system combinations.

A nonlinear contracting system has the following properties [15, 21, 22, 23]

- global exponential convergence and stability are guaranteed
- convergence rates can be explicitly computed as eigenvalues of well-defined symmetric matrices
- robustness to variations in dynamics can be easily quantified

### 2.1 Basic results

Our general dynamical systems will be in  $\mathbb{R}^n$ , deterministic, with  $f$  a smooth non linear function.

$$\dot{x} = f(x, t) \tag{1}$$

The basic theorem of contraction analysis, derived in [15], can be stated as:

**Theorem 1** (Contraction). *Denote the Jacobian matrix of  $f$  with respect to its first variable by  $\frac{\partial f}{\partial x}$ . If there exists a square matrix  $\Theta(x, t)$  such that  $\Theta(x, t)^T \Theta(x, t)$  is uniformly positive definite and the matrix*

$$F = \left( \dot{\Theta} + \Theta \frac{\partial f}{\partial x} \right) \Theta^{-1}$$

*is uniformly negative definite, then all system trajectories converge exponentially to a single trajectory, with convergence rate  $|\sup_{x,t} \lambda_{\max}(F)| > 0$ . The system is said to be contracting,  $F$  is called its generalized Jacobian, and  $\theta(x, t)^T \Theta(x, t)$  its contraction metric.*

It can be shown conversely that the existence of a uniformly positive definite metric

$$M(x, t) = \Theta(x, t)^T \Theta(x, t)$$

with respect to which the system is contracting is also a necessary condition for global exponential convergence of trajectories [15]. Furthermore, all transformations  $\Theta$  corresponding to the same  $M$  lead to the same eigenvalues for the symmetric part  $F_s$  of  $F$  [22], and thus to the same contraction rate  $|\sup_{x,t} \lambda_{\max}(F_s)|$ .

*Remark 2.1.* In the linear time-invariant case, a system is globally contracting if and only if it is strictly stable, and  $F$  can be chosen as a normal Jordan form of the system with  $\Theta$  the coordinate transformation to that form [15].

*Remark 2.2.* Contraction analysis can also be derived for discrete-time systems and for classes of hybrid systems [21].

Finally, it can be shown that contraction is preserved through basic system combinations, such as parallel combinations, hierarchies, and certain types of negative feedback, see [15] for details.

## 2.2 Contraction toward a linear subspace

The main theorem of [24] gives us the ability to prove contraction of all solutions to a subspace  $\mathcal{M}$  of the state space. It is a powerful tool that we will use in the symmetry studies, and can be stated as

**Theorem 2.** *Consider a linear flow-invariant subspace  $\mathcal{M}$  of the system ( $f(\mathcal{M}) \subset \mathcal{M}$ ) and the associated orthonormal projection matrix  $U^T U$  (we have  $V^T V + U^T U = I_n$  and  $x \in \mathcal{M} \iff Vx = 0$ ). All trajectories of the system converge exponentially to  $\mathcal{M}$  if the system*

$$\dot{y} = Vf(V^T y, t) \tag{2}$$

is contracting with respect to a constant metric. If furthermore we denote the contraction rate for (2) by  $\lambda > 0$ , then the convergence to  $\mathcal{M}$  will be exponential with rate  $\lambda$ .

We will call the above condition,  $V^T f V$  contracting for a constant metric, *contraction toward*  $\mathcal{M}$ .

*Remark 2.3.* The theorem uses mainly two independent hypotheses

- Contraction condition of Equation 2 :  $f$  contracts toward  $\mathcal{M}$
- Invariance condition of  $\mathcal{M}$  :  $f(\mathcal{M}) \subset \mathcal{M}$

## 2.3 Contraction yields robustness

It can be shown that (see section 3.7 in [15] for a proof and generalization)

**Theorem 3** (Contraction and robustness). *Consider a contracting system  $\dot{x} = f(x, t)$ , with a constant metric  $\Theta$  and contraction rate  $\lambda$ . Let  $P_1(t)$  be a trajectory of the system, and let  $P_2(t)$  be a trajectory of the disturbed system*

$$\dot{x} = f(x, t) + d(x, t)$$

*Then the distance  $R(t)$  between  $P_1(t)$  and  $P_2(t)$  verifies  $R(t) \leq \sup_{x,t} \|d(x, t)\|/\lambda$  after exponential transients of rate  $\lambda$ .*

## 3 Symmetries and contraction

The symmetries of a neural network, defined in a broad sense, can reflect important properties. There are many different ways to express symmetries, such as symmetry of the input, the output, or the system, all of which are usually interdependent.

### 3.1 Generic $\gamma$ operator

Consider a dynamical system  $\dot{x} = f(x, t)$ . A linear operator  $\gamma$  acting over the state space defines two usual “symmetries” :

- symmetry of the system state: if  $x = \gamma x$ , we will say that  $x$  is  $\gamma$ -symmetric

- symmetry of the dynamical system : if  $\gamma f = f\gamma$ , we will say that  $f$  is  $\gamma$ -equivariant [17]

Note that any linear operator belongs to  $GL$ , the general linear group, see for example [16] also using linear operators as symmetries.

The following simple result shows that a contracting dynamical system “transfers” its symmetries to its state trajectories.

**Lemma 3.1.** *If  $f$  is  $\gamma$ -equivariant and contracting, all solutions converge exponentially to a unique  $\gamma$ -symmetric trajectory  $x(t)$ .*

*Proof.* Since the system is contracting all solutions converge exponentially to a single solution  $x(t)$ . But  $\gamma x(t)$  is also a solution :

$$\frac{d}{dt}(\gamma x(t)) = \gamma \dot{x}(t) = \gamma f(x, t) = f(\gamma x, t)$$

hence  $x(t) \rightarrow \gamma x(t)$  exponentially. □

### A simple example: permutations

Let us illustrate  $\gamma$ -symmetry in the simple discrete case of a permutation operator.

Consider  $x \in E = \mathbb{R}^n$ , and write it as  $(x_1, x_2, \dots, x_n)$ , the action of a permutation  $\gamma$  on  $E$  is defined by  $\gamma x = (x_{\gamma(1)}, \dots, x_{\gamma(n)})$ .

Decompose  $\gamma$  into disjoint non-trivial cycles,

$$\gamma = \sigma_0 \circ \sigma_1 \dots \sigma_p$$

and decompose the space accordingly as

$$E = \mathbb{R}^n = E_{\sigma_0} \times E_{\sigma_1} \times \dots \times E_{\sigma_p} \times E_{I_q}$$

with  $E_{\sigma_i}$  the space of action of  $\sigma_i$ .

$\gamma$  symmetry of the state space describes *concurrent synchronization*: in each subspace  $E_{\sigma_i}$  the solution is synchronous, thus yielding  $p$  co-existing synchronous assemblies, as illustrated in Figure 1.

$$\begin{aligned}
\gamma &= (01)(23)(4) \\
E &= E_{(01)} \times E_{(23)} \times \mathbb{R} \\
x &= (x_0, x_1, x_2, x_3, x_4) \\
\gamma x &= (x_1, x_0, x_3, x_2, x_4)
\end{aligned}$$

We have  $f$   $\gamma$ -equivariant, indeed  $\gamma f = f \gamma$  :

$$\begin{aligned}
f_2(x_0, x_1, x_2, x_3, x_4) &= f_1(x_1, x_0, x_3, x_2, x_4) \\
f_1(x_0, x_1, x_2, x_3, x_4) &= f_2(x_1, x_0, x_3, x_2, x_4) \\
f_4(x_0, x_1, x_2, x_3, x_4) &= f_3(x_1, x_0, x_3, x_2, x_4) \\
f_3(x_0, x_1, x_2, x_3, x_4) &= f_4(x_1, x_0, x_3, x_2, x_4) \\
f_5(x_0, x_1, x_2, x_3, x_4) &= f_5(x_1, x_0, x_3, x_2, x_4)
\end{aligned}$$

And  $\gamma$  symmetry of  $x$  would be synchrony of  $x_0$  with  $x_1$  and  $x_2$  with  $x_3$  :

$$\left. \begin{array}{l} x_0 = x_1 \\ x_2 = x_3 \end{array} \right\} \iff x = \gamma x$$

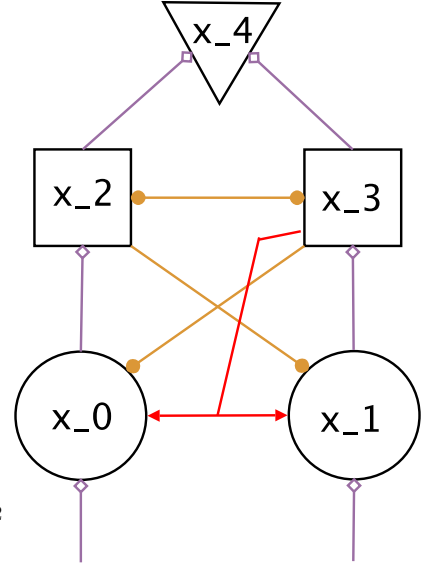


Figure 1: Toy example which could model a three layered network.

### 3.2 Spatio-temporal symmetries

A straightforward extension of spatial symmetries are *spatio-temporal symmetries*. Inspired by the theory developed by Golubitsky et al. [17, 25, 26, 27], we define a *spatio-temporal symmetry*  $h = (\gamma, T)$  using a spatial symmetry  $\gamma$  and a period  $T$ , according to

$$hx(t) = \gamma x(t + T)$$

Unlike the  $H/K$  theorem of Golubitsky et al. [25, 27], we do not restrict ourselves to permutation for the spatial symmetry, but, to some linear operator  $\gamma$  such as there exists an integer  $p_\gamma$ , classically called the order of  $\gamma$ , such that  $\gamma^{p_\gamma} = Id$ .

**Example :** Consider a 3-ring with the  $h$ -symmetry :

$$\gamma \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(x_1 + x_3) \\ x_2 \\ \frac{1}{\sqrt{2}}(x_3 - x_1) \end{pmatrix} \quad \text{so} \quad \gamma^2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_2 \\ -x_1 \end{pmatrix} \quad \text{and} \quad (\gamma^2)^4 = id$$

$x$  is  $h$ -symmetric when :

$$hx(t) = x(t) \iff \begin{cases} x_1(t) = \frac{1}{\sqrt{2}} (x_1(t+T) + x_3(t+T)) \\ x_2(t) = x_2(t+T) \\ x_3(t) = \frac{1}{\sqrt{2}} (x_3(t+T) - x_1(t+T)) \end{cases}$$

There is a strong interaction between  $x_3$  and  $x_1$  but interestingly  $hx(t) = x(t) \implies x_1(t) = x_1(t+8T)$  and for  $x_1(0) \neq 0, x_1(t) \neq x_1(t+T) \neq$  the same holds for  $x_3$  but  $x_2(t) = x_2(t+T)$ . This shows that two different rhythms have to coexist.  $\square$

An associated definition can be given for a dynamical system. Specifically, we will say that the system  $\dot{x} = f(x, t)$  (or its dynamics  $f$ ) is  $h$ -equivariant if

$$f(\gamma x(t), t) = \gamma f(x(t), t+T) \quad (3)$$

We then obtain for spatio-temporal symmetries a result similar to Theorem 3.1, describing the transfer of symmetries from system dynamics to system trajectories.

**Theorem 4.** *If  $f$  is contracting and  $h$ -equivariant, then after transients the solutions are  $p_\gamma T$  periodic and exhibit the spatio-temporal symmetry  $h$ .*

Basically, all the solutions tend to a periodic solution  $x_p(t)$  with the symmetry  $h : x_p(t) = \gamma x_p(t+T)$ . This result is an extension of the result that a periodic contracting system exhibits a unique solution of same period [15].

*Proof.* If  $x(t)$  is a solution, then  $\gamma x(t+T)$  is also a solution:

$$\frac{d}{dt}(\gamma x(t+T)) = \gamma \dot{x}(t+T) = \gamma f(x(t+T), t+T) = f(\gamma x(t+T), t)$$

Thus, since  $f$  is contracting,  $x(t) \rightarrow \gamma x(t+T)$  exponentially fast. This in turn shows that the solution tends to a periodic signal exponentially : by recursion

$$x(t) \rightarrow \gamma^{p_\gamma} x(t + p_\gamma T) = x(t + p_\gamma T)$$

so that  $\forall t \in [0; p_\gamma T]$   $x(t + np_\gamma T)$  is a Cauchy sequence, and therefore the limiting function  $\lim_{n \rightarrow \infty} x(t + np_\gamma T)$  exists, which completes the proof.  $\square$

### 3.3 Spatial symmetries accomodate weaker contraction

When dealing only with spatial symmetries, we can weaken the contraction condition on  $f$  while still transferring the symmetries of  $f$  to the system trajectory.

Consider the linear subspace of  $\gamma$ -symmetric states,  $\mathcal{M}_\gamma = \{x, x = \gamma x\}$ . Recall first a standard result linking the symmetry of the system to this linear subspace:

**Lemma 4.1.**  *$f$  is  $\gamma$  equivariant  $\implies \mathcal{M}_\gamma$  flow invariant*

*Proof.* if  $x \in \mathcal{M}_\gamma$  and  $f$  is  $\gamma$ -equivariant :  $\dot{x} = f(x) = f(\gamma x) = \gamma f(x) = \gamma \dot{x}$  □

In this context, we can thus write Theorem 2 as

**Lemma 4.2.** *If  $\mathcal{M}_\gamma$  is flow-invariant by  $f$  (or sufficiently, by Lemma 4.1, if  $f$  is  $\gamma$ -equivariant) and  $f$  contracts toward  $\mathcal{M}_\gamma$ , then all solutions  $x(t)$  converge exponentially to  $\gamma$ -symmetric trajectories.*

We will denote by  $V_\gamma^T V_\gamma$  the orthogonal projector on  $\mathcal{M}_\gamma^\perp$ .

As these lemma shows, the more generic  $\gamma$  is, the stronger the contraction condition. In the generic lemma 3.1 the hypotheses of symmetry and contraction are *independent* on the contrary to last lemma 4.2 where the contraction condition depends explicitly on  $\gamma$ .

This link between symmetry and contraction is not particularly convenient, since later in section 5 we will aim to shape the equivariance of  $f$ , and thus the symmetries of the output, while preserving sufficient contraction properties. We now show how to avoid this direct dependence.

### 3.4 $\mathcal{M}_{all}$ and $\mathcal{M}_\Gamma$ spaces

We show how to strengthen the contraction condition of lemma 4.1 to make it independent of the symmetry condition, or at least allowing to have the theorem hold with a set  $\Gamma$  of different symmetries.

First note:

**Lemma 4.3.** *The choice of  $V$  to represent the orthogonal projector has no effect on the contraction toward  $\mathcal{M}^\perp$ .*

See proof in appendix 7.6.2. The proof also shows that, by contrast, using a non orthogonal projector  $V_2 = TV$  with  $T$  square invertible is not sufficient in general.



*Remark 3.1.* The above lemma shows that the contraction condition toward a subspace is preserved when applying an orthonormal transformation  $M$  to the projector  $V$ . Moreover one can also trivially change the metric with an orthonormal transformation :

$$M\Theta VfV^T\Theta^{-1}M^T < 0 \iff \Theta VfV^T\Theta^{-1} < 0 \iff \Theta MVfV^TM^T\Theta^{-1} < 0$$

Now the main result of this section :

**Lemma 4.4.** *Consider two linear subspaces  $\mathcal{M}$  and  $\mathcal{M}_2$ , with  $\mathcal{M} \subset \mathcal{M}_2$ . If  $f$  contracts toward  $\mathcal{M}$ , then  $f$  contracts toward  $\mathcal{M}_2$ .*

*More precisely, let  $V^TV$  (resp.  $V_2^TV_2$ ) be an orthogonal projector onto  $\mathcal{M}^\perp$  (resp.  $\mathcal{M}_2^\perp$ ). If  $f$  contracts toward  $\mathcal{M}$  with constant metric  $\Theta$ , then taking  $U^T$  a partial isometry from  $E_{\mathcal{M}_2^\perp}$  to  $E_{\mathcal{M}^\perp}$ , with kernel 0 and image  $\text{Im}(\Theta VV_2^TE_{\mathcal{M}_2})$ ,  $f$  contracts toward  $\mathcal{M}_2$  with constant metric  $\Theta_2 = U\Theta VV_2^T$*

See proof and definitions in appendix (7.6.1)

*Remark 3.2.* One would be tempted to set  $U^T = \Theta VV_2^T$  but this is possible only when  $\Theta$  itself is an orthonormal transformation. In particular when  $\Theta = Id$  we can set  $U^T = VV_2^T$  giving us  $\Theta_2 = Id$

The main consequence of lemma 4.4 is the ability to determine a sufficient contraction condition for a set  $\Gamma$  of symmetries. Indeed we just showed that a sufficient contraction condition would be the contraction toward  $\mathcal{M}_\Gamma = \bigcap_{\gamma \in \Gamma} \mathcal{M}_\gamma$ . This condition and the equivariance with respect to a specific  $\gamma$  allow lemma 4.2 to be applied to this  $\gamma$ .

In the generic linear case, there is no non trivial unifier, since the intersection of all linear subspaces is reduced to  $\{0\}$ , corresponding to the contraction of the full system.

But when considering the permutations, the common subspace exists :  $\mathcal{M}_{all} = \{\forall i, \forall j, x_i = x_j\}$ . This subspace of full synchrony will be very handy and may play an important role in neural networks.

Moreover the contraction toward  $\mathcal{M}_{all}$  will be quite easy to prove when using eventual symmetry of the system. To have more insight into the kind of computation, see Section 7.3.

In summary to the contraction and symmetry section, the main theorem we will use can be stated as

**Theorem 5.** *Consider a  $\gamma$ -equivariant or with  $\mathcal{M}_\gamma$  flow-invariant system, and assume one of the following contraction properties (sorted by decreasing strength)*

- contraction of the system
- contraction toward  $\mathcal{M}_\Gamma$  with  $\gamma \in \Gamma$
- with  $\gamma$  a permutation, contraction toward  $\mathcal{M}_{all}$

Then all solutions converge exponentially to a  $\gamma$ -symmetric trajectory  $x(t)$ .

*Proof.* We only apply Lemma 4.2 and 3.1. The only change is the generalization in the fully contracting case by using  $\mathcal{M}_\gamma$  flow-invariance hypothesis : a contracting system has a unique solution independent of the initial conditions, then taking a trajectory beginning in  $\mathcal{M}_\gamma$  will stay in  $\mathcal{M}_\gamma$  by flow-invariance thus forcing the unique solution to be in  $\mathcal{M}_\gamma$ .  $\square$

*Remark 3.3.* All the symmetries of  $f$ , complying the theorem, will be transferred to the trajectories. *The system will maximize synchrony* : for example, if  $f$  complies the theorem with  $\gamma = (0, 1)$  and  $\gamma = (1, 2)$  then the system will lead to  $x_0 = x_1 = x_2$ .

## 4 Input continuity

### 4.1 Input continuity motivations

We now exploit these results on global symmetries in a control framework, by introducing control inputs in the system to modulate its dynamics.

We will observe  $f$  under the influence of some input  $u$ :

$$\dot{x}(t) = g(x, u(t), t) = f(x, t) \quad (4)$$

The input  $u$  doesn't represent all the input of the actual system, the rest of the actual input can still be hidden as before inside the function  $g$ . This choice underlines the fact that we want to study the system response to a decisive part  $u$  of the actual input. Then the idea is to control the output with the input to get the property that we want see Section 5. In an in vivo situation, we can see all the feedback loops from the upper part of the brain to the bottom part of the brain as control inputs  $u$ , but also the input from the bottom to the top, and every other connections.

The response to  $u$  of the system will be some output  $s$  defined by the state of the system. Typically we will use  $s(t) = x(t)$  or some projection of the state  $s(t) = Px(t)$ .

With this point of view we introduce *input continuity* analysis, in order to describe the properties of a system's output knowing the properties of its input while ensuring stability and robustness.

As a general matter, we want to be able to say two things, first “if  $u$  has this property then  $s$  will have that property” and secondly “if the input  $u$  is close to an ideal input  $u'$  then  $s$  will be close to the ideal output  $s'$ ”. This will be formalized by the notion of “input continuity” :

$$\forall \epsilon \geq 0, \exists \eta \geq 0 \text{ such as } d_1^t(u', u) \leq \eta \implies d_2^t(s', s) \leq \epsilon \quad (5)$$

$d_1^t$  (resp.  $d_2^t$ ) is a pseudo distance of the space of the input  $u$  (resp. the output  $s$ ). This pseudo distances help us to define the notion of being close as traditionally but also the properties corresponding to the chosen pseudo distance : if  $d_1^t(u', u) = 0$ , then  $u$  and  $u'$  are in the same class defining some property see Section 7.2.2 for details with some interesting examples shown in Section 7.2.3.

*Depending greatly on the distance we use, input continuity will be a modular tool to ensure robustness of specific properties of the output given the properties of the input.*

The modularity of input continuous block is something very generic and powerful, more discussion can be found in section 7.2.1, but from now we will restrict ourself to the study of a powerful input continuity found in contracting systems.

## 4.2 Input continuity and contraction

There is no generic way to show input continuity of a system. Depending on the type of system (discrete or continuous) and the distance we use, we would have to examine each case we encounter. But in the case of a contracting system, we can state some powerful generic properties. This will allow us to combine Theorem 5 and the input continuity without any effort, the contraction being already an hypothesis. The main tool to study input continuity of a contracting system is its robustness :

### 4.2.1 Contracting systems

We will compare the perturbed state  $s'$  with perturbed input  $u'$  and the wanted state  $s$  with perfect input  $u$  :

$$\begin{aligned} \dot{s} &= f(s, u, t) \\ \dot{s}' &= f(s', u', t) = f(s', u, t) + h(s', t) \\ \text{with } h(t) &= f(s', u', t) - f(s', u, t) \end{aligned}$$

Then, if  $f$  is contracting, we can prove with the generalized form of the robustness seen in [15] that

$$\dot{R} + \lambda R \leq \|h(t)\| \quad (6)$$

$$R(t) \leq e^{-\lambda(t-t_0)} R(t_0) + \int_{t_0}^t e^{-\lambda(t-\tau)} \|h(\tau)\| d\tau \quad (7)$$

with  $r = s' - s$ ,  $R(t) = \|r(t)\|$ ,  $\|\cdot\|$  the norm of the space in which  $f$  is proved contracting with contraction rate  $\lambda$ . By convention we will have  $R(-\infty) = 0$ .

*Remark 4.1.* If the contraction analysis uses a metric, it is reflected in the norm  $\|\cdot\|$ , for instance in the case of the use of the 2-norm  $\|\cdot\|_2$  and a metric  $\Theta$  we will use  $\|\cdot\| = \|\Theta \cdot\|_2$

We can prove the input continuity considering the space of the input signal and output signal with these two norms :

$$\|x\|_{\infty}^{-\infty,t} = \text{Sup}_{\tau < t} \{(\|x(\tau)\|)\} \quad (8)$$

$$N_{\alpha, \|\cdot\|}^{-\infty,t}(x) = \int_{-\infty}^t \|x(\tau)\| e^{-\alpha(t-\tau)} d\tau \quad \text{with } \alpha > 0 \quad (9)$$

**Theorem 6.** *If all the signals are bounded and  $f$  is contracting and uniformly continuous in time, then we have the input continuity using the uniform norm (8) in both input and output space.*

*Proof.* From 6 we have :  $\|r\|_{\infty}^{-\infty,t} \leq \frac{1}{\lambda} \|h\|_{\infty}^{-\infty,t}$ . In the mean time, contraction gives us space continuity, then using the hypothesis of uniform continuity in time and the Heine's theorem over the compact space of bounded signals,  $\exists k \in \mathbb{R}$ ,  $\|h\|_{\infty}^{-\infty,t} \leq k \|u' - u\|$ .  $\square$

Note that the boundedness of the input signals is a plausible condition *in vivo*.

In our control context, a more flexible and meaningful tool is the norm (9) with exponentially fast forgetting :

**Theorem 7.** *If all the signals are bounded and  $f$  is contracting and uniformly continuous in time, then we have the input continuity with  $N_{\alpha, \|\cdot\|}^{-\infty,t}$  in the input space and  $N_{\beta, \|\cdot\|}^{-\infty,t}$  in the output space.*

See proof in appendix 7.4.1.

*Remark 4.2.* When dealing with this kind of norms, Lemma 10.2 can be very convenient.

### 4.2.2 Systems contracting toward a subspace

With contraction toward linear subspace, we want to ensure the property "we are in  $\mathcal{M}$ " which can be easily characterized with a generic semi-norm see Section 7.2.2 for more explanations :

$$N_{\mathcal{M}}(x) = \|Vx\|$$

We can now state an adapted version of Theorem 7

**Theorem 8.** *If all the signals are bounded,  $f$  uniformly (in time) continuous, with Equation 2 contracting and  $\mathcal{M}$  flow-invariant, then the system is input continuous with input norm  $N_{\alpha, \|\cdot\|}^{-\infty, t}$  and output pseudo norm  $N_{\alpha, N_{\mathcal{M}}}^{-\infty, t}$ .*

*Remark 4.3.* This obviously works with the uniform norm in the same way.

*Proof.* We can apply Theorem (7) on the contracting system (10) with space norm  $\|\cdot\|$ , giving us input continuity of this system with norm  $N_{\alpha, \|\cdot\|}^{-\infty, t}$  ( resp.  $N_{\beta, \|\cdot\|}^{-\infty, t}$  ) as input norm ( resp. output norm ). Using notation of the Theorem, we can link  $y$  and  $x$ , indeed if we set  $y = Vx$  we get the system

$$\dot{y} = V\dot{x} = Vf(x) = Vf(V^T y + U^T Ux) \quad (10)$$

which is contracting with respect to  $y$ , equivalently with the contraction of the system (2). Then we have  $N_{\mathcal{M}}(x) = \|Vx\| = \|y\|$  so we can directly apply the result to the original system using  $N_{\alpha, N_{\mathcal{M}}}^{-\infty, t}$  as output norm.  $\square$

## 5 Control

With the power given by theorem 5 and the flexibility given by input continuity, we can now get the system to exhibit specific symmetries with the help of a small controlling input. The global property of contraction and input continuity will be required to robustly do transient and multiple changes in the system and the symmetries of the output.

### 5.1 Main idea

Rather than looking at symmetrical solutions a system may exhibit, as in the H/K Theorem of Golubitsky et al. [17, 25, 26, 27], we consider what symmetries the system may exhibit when submitted to specific external inputs.

We consider a system of the form

$$\dot{x} = f(x, t) = g(x, u(t), t)$$

where now  $u(t)$  is a “control input”. We will control the symmetry of the system’s output by modifying its input. To do so we will use the theorems 5 and 4 on the function  $f$ . We will need

- a symmetry condition ( $\gamma$ -equivariance or flow-invariance)
- a contraction condition (contraction or contraction toward a subspace)

The contraction condition (in any form) will give us input continuity as shown in section 4.2, allowing us to plug the input at any time instead of controlling the system from the beginning, and still be exponentially close to the desired output. The symmetry condition will lead the system to a state expressing the desired symmetry.

The input can have different functions. It can determine the contraction condition as explained in the following 5.3, but also, and mainly, change the symmetries of the system, as we now detail.

## 5.2 Selection of spatio-temporal symmetries of the system

We first need to link the symmetries of  $f$  and those of  $g$  and  $u$ .  $g$  will be said  $h$ -equivariant if

$$g(\gamma x, \gamma y, t) = \gamma g(x, y, t + T)$$

**Theorem 9.**  $g$   $h$ -equivariant and  $u$   $h$ -symmetric  $\implies f$   $h$ -equivariant

*Proof.*

$$\begin{aligned} f(\gamma x(t), t) &= g(\gamma x(t), u(t), t) = g(\gamma x(t), \gamma u(t + T), t) \quad \text{by symmetry of } u \\ \gamma f(x(t), t + T) &= \gamma g(x(t), u(t + T), t + T) = g(\gamma x(t), \gamma u(t + T), t) \quad \text{by equivariance of } g \end{aligned}$$

□

In the general case,  $g$   $h$ -equivariant is not sufficient, moreover, increasing the symmetry thanks to the input is very unlikely, since it would require an intelligent input, quite as complex as the neuron model. This is not our goal since we consider the input to be “small”, and the neuron model a realistic non linear dynamical system. Thus for practical purposes the theorem is an equivalence. Once the symmetries  $\Gamma_g$  of  $g$  are determined, we can set an input with the symmetries  $\Gamma_u$  to control

the final symmetries  $\Gamma$  by considering  $\Gamma = \Gamma_g \cap \Gamma_u$ , i.e., intersecting the symmetries of  $g$  with those we set in the input.

**Example :** Consider the simple dynamics

$$f(x, t) = g(x, u(t), t) = \begin{cases} -x_1^3 + u_1(t) + \sin(t) \\ -x_2^3 + u_2(t) + \sin(t + \pi/3) \\ -x_3^3 + u_3(t) + \sin(t + 2\pi/3) \end{cases}$$

Here  $g$  is  $h = ((1, 2, 3), \pi/3)$ -equivariant, thus also  $h^2 = ((1, 3, 2), 2\pi/3)$ -equivariant and  $h^3 = (id, 2\pi)$ -equivariant, etc. Taking for example  $u$   $h^2$ -symmetric, but not  $h$ -symmetric, we have  $f$   $h^2$ -equivariant but not  $h$ -equivariant. Note that then the solution will be  $6\pi$  periodic instead of  $2\pi$  periodic, and also that nothing is needed or proved about the respective phases of the different signals and elements.

This also shows that the creation of symmetries is unlikely. If  $g$  was only  $h^2$ -equivariant, having  $u$   $h$ -symmetric will not make  $f$   $h$ -equivariant, but only  $h^2$ -equivariant.  $\square$

### 5.3 Control of the contraction condition of the system

Having the system always contracting will probably not be the generic case and the most biologically sound. Rather, we want to “turn on” the contraction property at a specific time using the input. It can be represented by :

$$\dot{x} = f(x, t) - k\chi_{on}x \tag{11}$$

Having  $k$  big enough and the activator  $\chi_{on} = 1$ , the system will contract. With some systems like a set of FitzHugh-Nagumo elements, putting such a term only over the potential variables will give contraction toward  $\mathcal{M}_{all}$ .

This “contracting input” is a transient negative feedback loop which can be turned on and off through the control of the activator. This can’t be as simple in neural models, but we propose a quite meaningful and simple “implementation” :

We consider the circuit of Figure 9 but without delay  $d$ . This circuit seems to be part of the neighborhood of many cortical pyramidal cells in the treatment of extracortical afferent excitations [28]. We suggest a behavior : The pyramidal neuron  $x_p$  would have lots of gap junction with its touching inhibitory interneuron  $x_i$ . The interneuron being way smaller will be driven by the pyramidal one, so that  $x_i \simeq x_p$ . Next setting a low firing threshold for the interneuron would allow to have it spiking proportionally to its potential. The resulting inhibition of the pyramidal

neuron will thus have the desired shape  $\simeq -k.x_p$ . Input continuity permit to compute the distance between this implementation and the perfect instantaneous negative feedback.

The activator will then be easily implemented by some inhibition of the interneuron : no inhibition means  $\chi_{on} = 1$ , 0 otherwise.

The first two examples use the activated contraction to control spatio-temporal symmetries (example of section 5.4.2 or section 5.4.1). Then we show in more details a grid example using the  $\mathcal{M}_{all}$  idea.

## 5.4 Examples

Throughout this section we illustrate some of the above possibilities shown through basic examples, mostly using FitzHugh-Nagumo neural models,

$$\begin{cases} \dot{v} = v(\alpha - v)(v - 1) - w + I \\ \dot{w} = \beta(v - \gamma w) \end{cases} \quad (12)$$

$$\alpha = 6, \beta = 3, \gamma = 0.03$$

with  $I$  the synaptic input function. Although most of the time we refer to our system elements as “neurons”, one should notice that more generally the theory developed here applies to  $f$  representing neural networks, whose equations can be actually very similar to FitzHugh-Nagumo models.

The use of FitzHugh-Nagumo neural models is motivated by its simplicity while still a reasonably descriptive neuron model, and it has the desired properties of contraction toward  $\mathcal{M}_{all}$  when coupled only through the potential variable see [24]. This property is kept when using the more precise Hodgkin-Huxley model see [29].

### 5.4.1 Leading to unstable state : transient synchronization

In this example we will use the action of a contracting input making the system contracts toward  $\mathcal{M}_{all}$  from time 75 to 95. This transient contraction results in a transient synchronization, which is often considered as a very important neural processing process [30, 31]. Consider the system seen in Figure 2. Neuron 1 and 2 are two FitzHugh-Nagumo neurons with an inhibitory symmetrical link between them. As we can see Figure 3 before we put the contracting input, the mutual inhibitory link leads naturally the system to antiphase. But after synchronizing the two neurons by force with the input, they stay in the unstable state where they are equal see Figure 3a, this during transient when some level of noise is added see Figure 3b.



This example illustrates the idea that with the input we can lead the system to a non 'natural' state, in much more complex networks this could be some basic phenomenon to allow different computations with the same network.

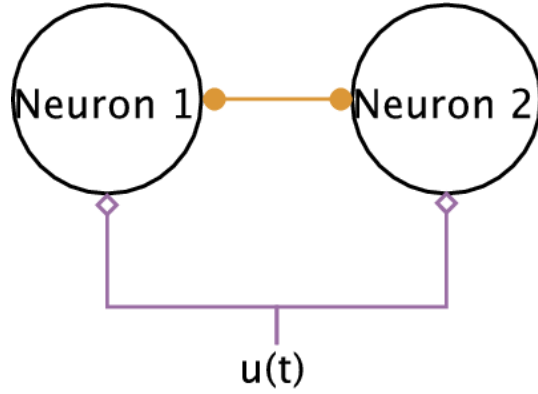
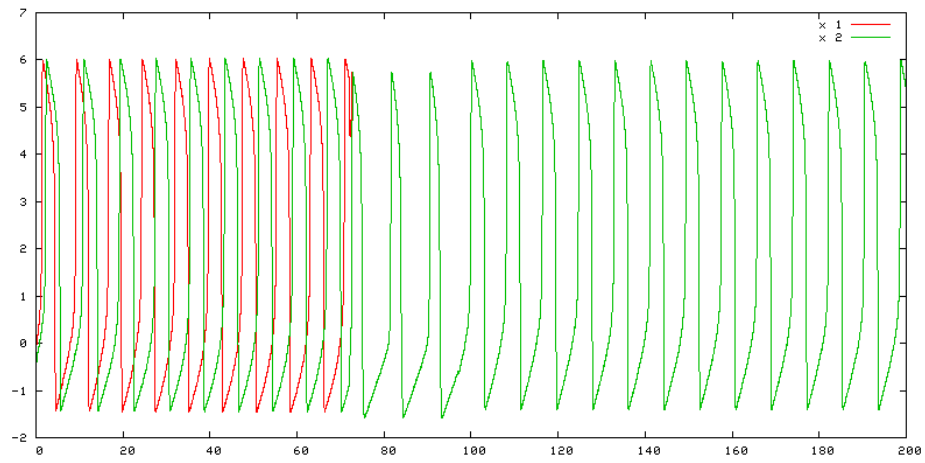
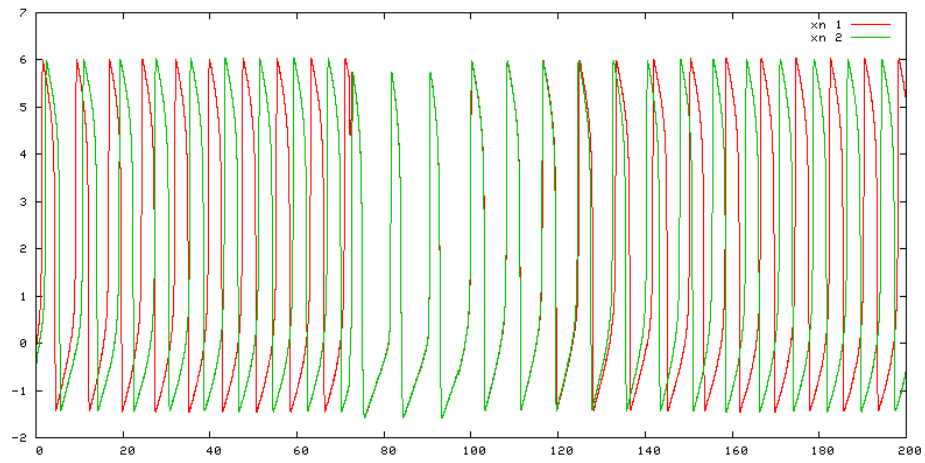


Figure 2: System for transient synchronization

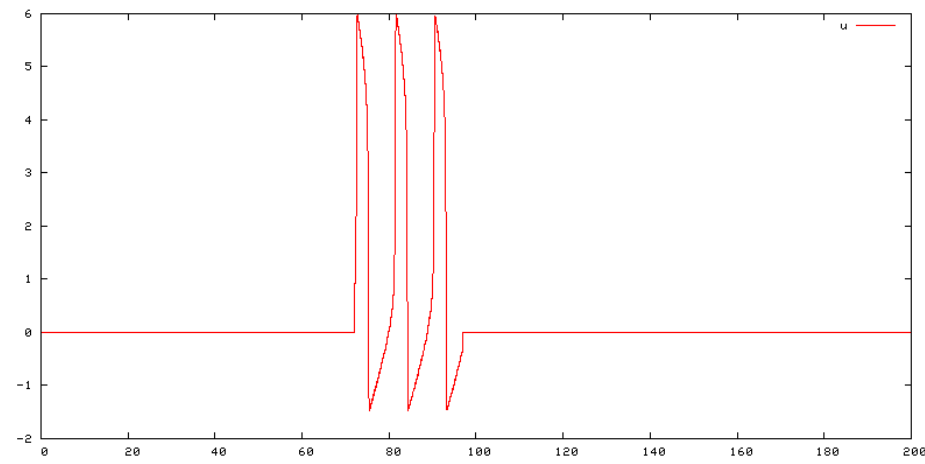
$$\begin{aligned} \dot{x}_1 &= f(x_1) - \mu x_2 + e + I_1 \\ \dot{x}_2 &= f(x_2) - \mu x_1 + e + I_2 \\ I_i &= \lambda \chi_{t \in [75, 95]}(u(t) - x_i) \end{aligned}$$



(a) without noise



(b) with noise



(c) input  $u$  with activator

Figure 3: Transient synchronization  $\mu = 0.1, \lambda = 5, noise = 5, e = 20$

### 5.4.2 Choose the spatio-temporal symmetry in a 3 ring hysteresis system

We have here Figure 4 a ring of 3 FitzHugh-Nagumo neurons, each inhibiting its right neighbor. This system as we can see Figure 5 has a stable state where none of the neurons spike (here  $e$  is not big enough to make them spike because of the overall inhibition) but also another stable state  $((1, 2, 3), T/3)$  symmetric, where the neurons are spiking one after the other and each neuron has a period  $T$  (the inhibition being in the refractory period of the next one, makes the FitzHugh-Nagumo spike shortly after). To pass from one state to the other we use a contracting input function with an input signal exhibiting the symmetry we want to see, in figure 5 we lead the system to the rotating wave and then back to silence. The system

$$\begin{aligned} \dot{x}_1 &= f(x_1) - \mu x_3 + e_1 + I_1 \\ \dot{x}_2 &= f(x_2) - \mu x_1 + e_2 + I_2 \\ \dot{x}_3 &= f(x_3) - \mu x_2 + e_3 + I_3 \\ I_i &= \lambda \chi_{on}(u_i(t) - x_i) \end{aligned}$$

*Remark 5.1.* The control input to get back to the silent mode is here a long step but can be any spatio-temporal identity signal ( equal for neuron 1, 2 and 3). Inspired from the visual saccades involving bursting, this long step could model a high frequency burst, being here a way to reinitialize our network to a silent state before a new computation cf [32].

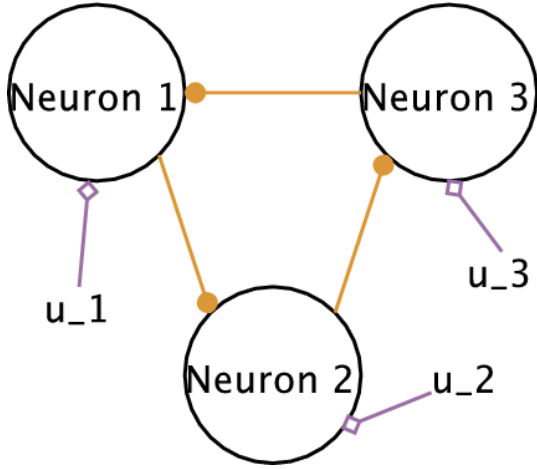


Figure 4: 3 ring hysteresis system

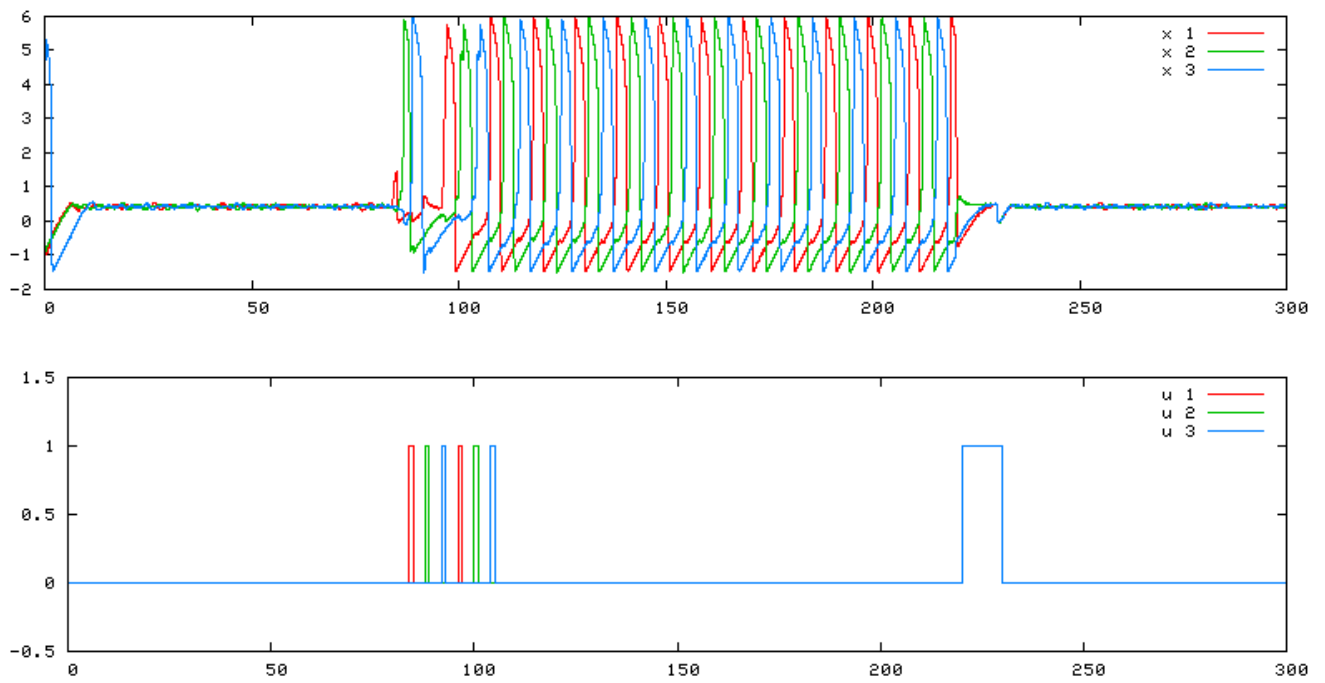


Figure 5: 3 ring hysteresis system, state selection by input

### 5.4.3 Grid and group selection

This example illustrate the idea of symmetry selection without changing the contraction condition ( see section 5.2) and illustrating a common issue of 2D segmentation. The concept will be to use a toric grid being contracting toward  $\mathcal{M}_{all}$ , then to select thanks to the input the desired (necessarily flow invariant) groups.

The grid is formed with neurons connected through diffusive connections ( representing gap junctions and other direct contacts between neurons ) to their four closest neighbors. We will set the coupling strength  $k$  to be strong enough to ensure contraction toward  $\mathcal{M}_{all}$  ( it exists see balanced coupling in [24]). Following the control idea of section 5.2, the system will polysynchronize depending on the flow invariant subspaces found in the input  $u$ .

Specifically we use a 5x5 grid of identical FN neurons modeled as (12). For each run, the initial conditions are set so that the neurons phase are spread out. We will consider 2 flow invariant patterns chosen among the one we can find in [33], namely pattern 1 and 2 of figure 6, the coloring represents the wanted flow invariant groups. *input\_0* ( resp. *input\_1* ) will represent the input of the white (resp. black) neurons. To separate the two groups of neurons but also showing some interesting interactions (we don't want all the neurons to be synchronized even if the FitzHugh-Nagumo model being a 2 dimensional model goes very easily to full synchrony with this grid connection) we chose (see plots in figure 7a)

$$\begin{aligned} input\_0i &= 2 \sin(2\pi t/60) + 21 \\ input\_1i &= 10 \sin(2\pi t/8) - 20 \end{aligned}$$

When we will say with noise we add to the input of each neuron a random noise taking a new value between 0 and 1 each 0.05 second.

We first apply pattern 1. Without coupling Figure 7b we observe the natural behavior of the FitzHugh-Nagumo model which 'synchronize' with its input quite easily as can be observed with one of the two groups. Setting  $k = 0.3$  Figure 7c gives the expected behavior, two groups appear exponentially fast.

We then use the input to change the synchronized groups from pattern 1 to pattern 2 at time  $t = 150$ . First as expected the speed of convergence increase with  $k$  but also the synchrony among groups, indeed the coupling tends to synchronize groups also.

Adding individual noise Figure 8 obviously prevent synchrony without the coupling, but with the coupling we obtain the desired grouping with some glitches allowed by the input continuity (the difference in the norms are always above a certain mean of the integral of the noises).

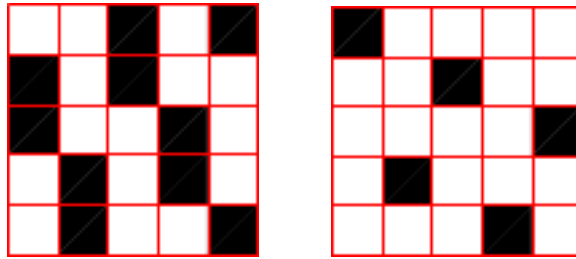
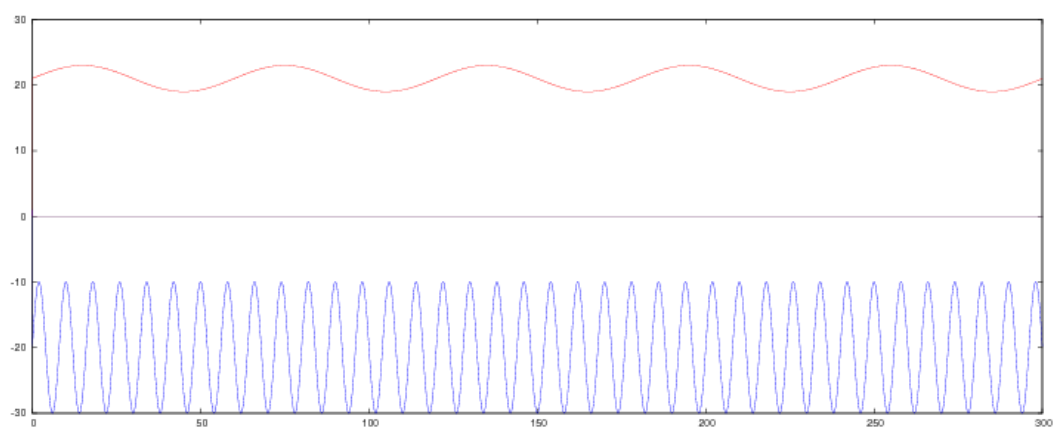
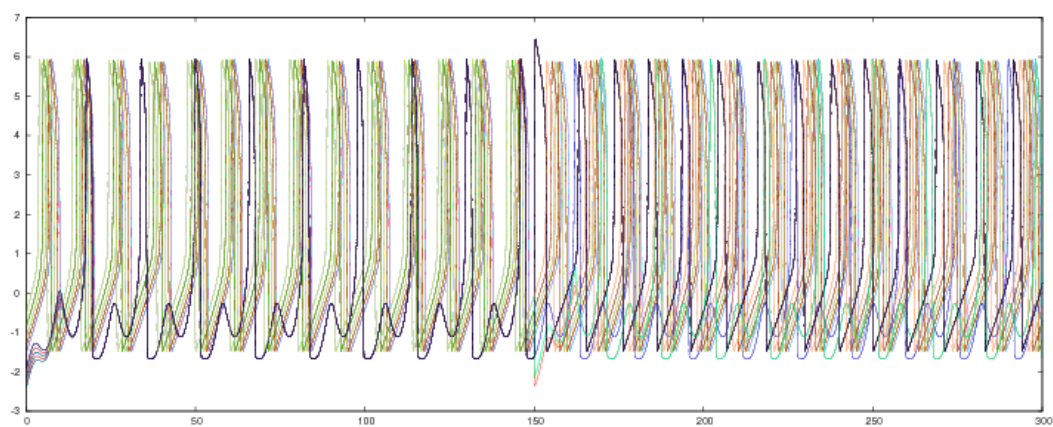


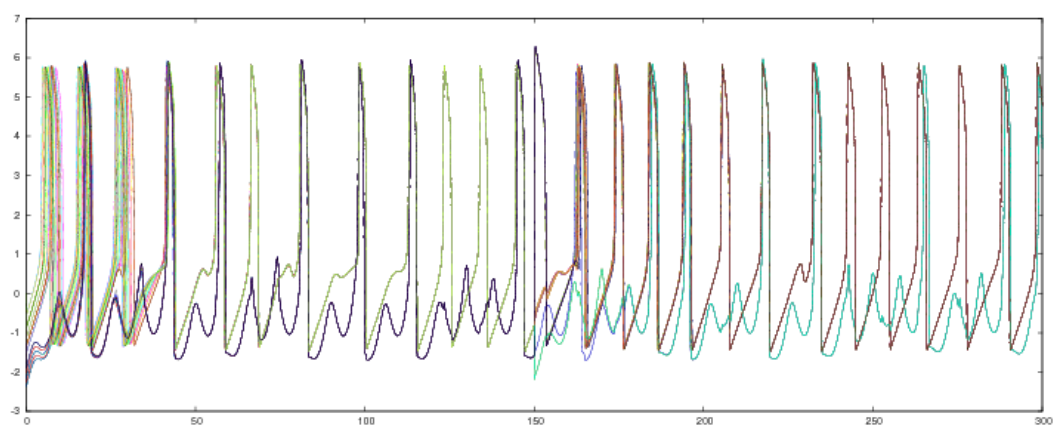
Figure 6: Grid pattern used to determine the groups



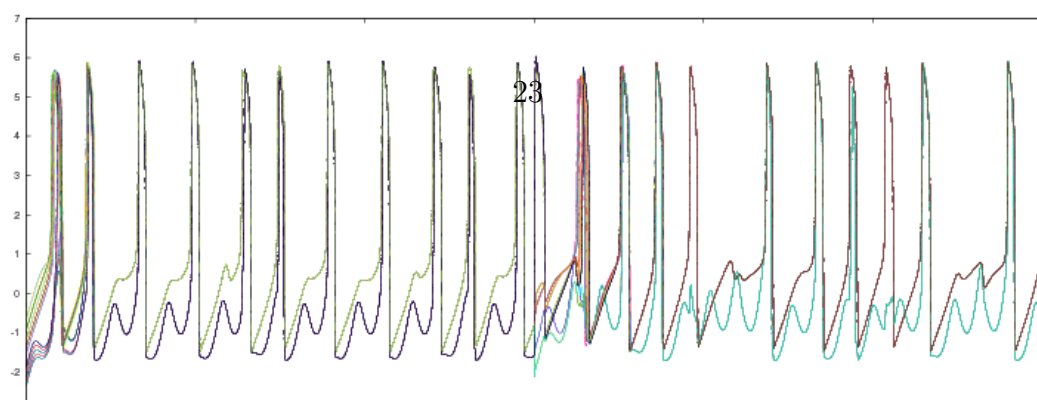
(a) Input

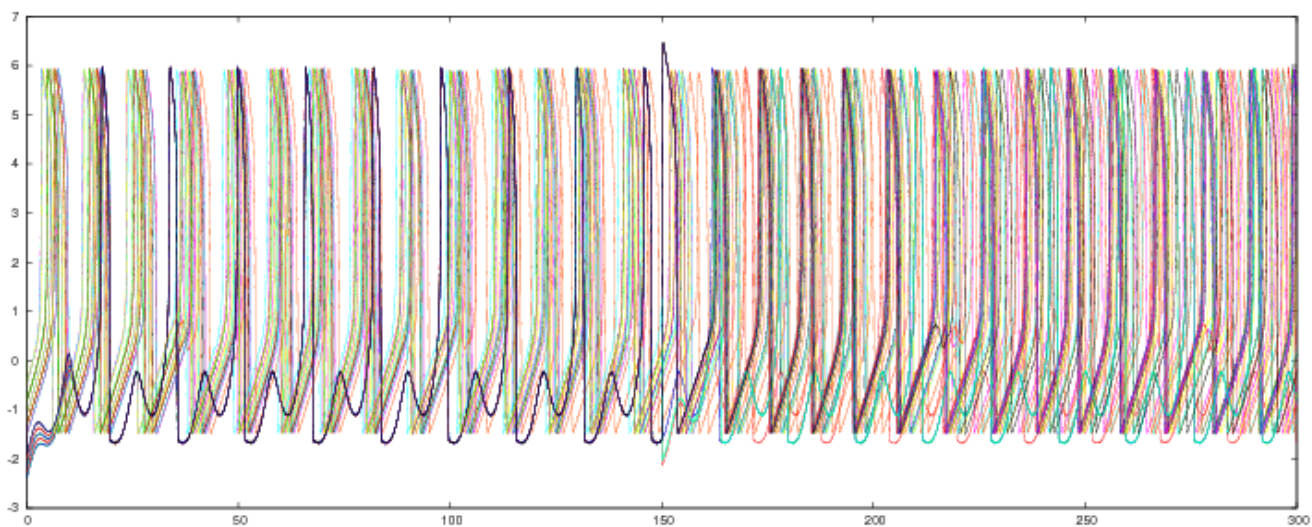


(b)  $k = 0$

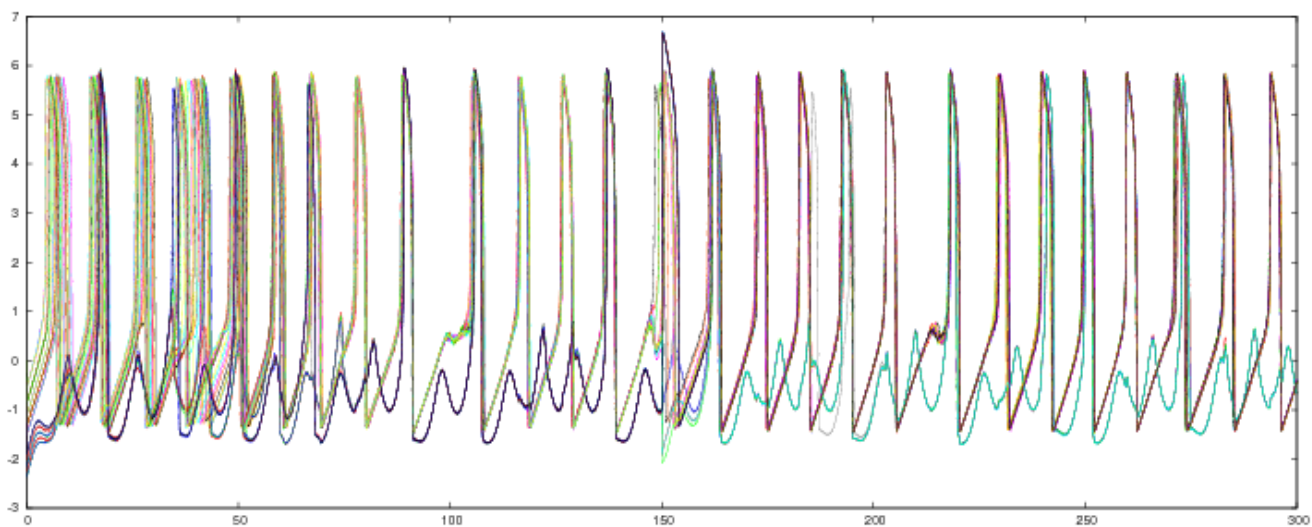


(c)  $k = 0.3$

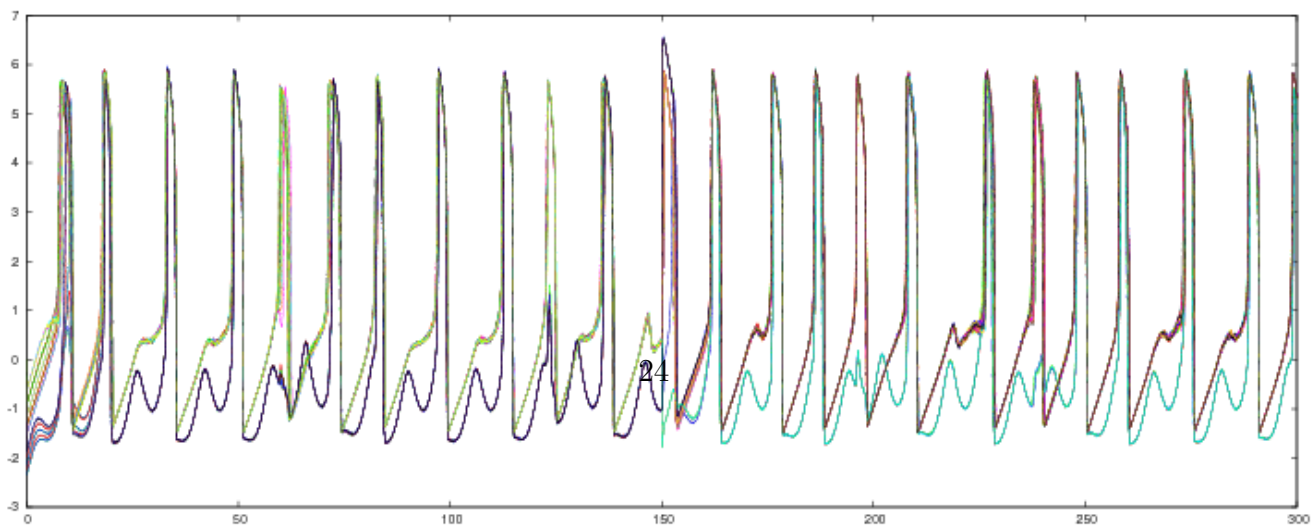




(a)  $k = 0$



(b)  $k = 0.3$



(c)  $k = 0.7$

Figure 8: Grid pattern 1 followed by pattern 2 at  $t = 150$  with noise



## 6 Discussion

Robustness and globalness are interesting properties of the studied methods. Many studies use approximations about the trajectories, considering that the neurons are close to their limit cycle, for instance [34, 35, 36]. Since contraction is a global property, nothing is assumed about the location of the state of the system when we plug in a new input, allowing fast input-driven switching between different synchronization patterns. Global exponential convergence to the desired behaviors is obtained, with quantifiable convergence rates. Globalness also avoids some of the topological difficulties associated to the study of large networks of phase oscillators. In the input continuity proof, only bounded signals are needed.

The modularity of the tools is a strong property allowing to mix studies of networks done at different scales (neuron, neural mass, neural assemblies and so on). Indeed, while we use neuron models as our main dynamical system unit, the development can be applied to other dynamical systems networks.

The symmetries used here are quite generic, the extension of spatial symmetries to linear operators and the extension to spatio-temporal symmetries seems important, since it is required to deal with the idea of spatio-temporal pattern coding in the brain, and natural external stimuli.

Two main weaknesses can be pointed out:

First our control over the symmetries of the system doesn't prevent the system to exhibit *more* symmetries in the end — mainly ensuring to have two synchronized groups of neurons doesn't prevent to have in fact total synchrony. To prevent the system to go to more synchrony, it is important in practice to actively separate the groups (as we did in the grid example), e.g. through inhibition, or break the symmetry.

Second, the spatio-temporal case is very interesting and quite unexplored. In this paper we only drew conclusions for fully contracting systems, a restrictive condition. A relaxed condition similar to the existence of  $\mathcal{M}_{all}$  in the spatial case would be more desirable (if perhaps unlikely).

Finally, the small circuit of a main neuron and its inhibitory interneuron is interesting in its own right. This circuit is proposed as a plausible implementation of “contracting inputs” in section 5.3, but also represents an frequency selector circuit as seen in example 7.1. Its biological relevance may be further investigated.

## 7 Appendix

### 7.1 Frequency selector / contraction activator network

The small classical cortical circuit : Figure 9 seems to be omnipresent on most cortical cells, set to treat the extracortical afferent excitations. [28] We have already seen that this network could be of great use to control the contraction of the system Section 5.3. But with different parameters it can be an interesting frequency selector.

We will use the fact that we can control the frequency of both neurons with the input (by synchronizing the neuron with the input), conjugated with a fixed delay of inter-inhibition which will be the intrinsic frequency shut down of this circuit :

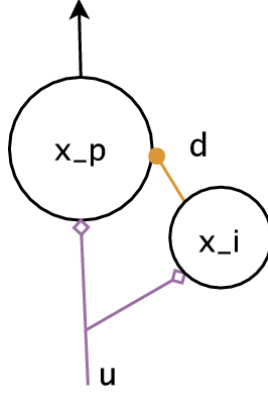
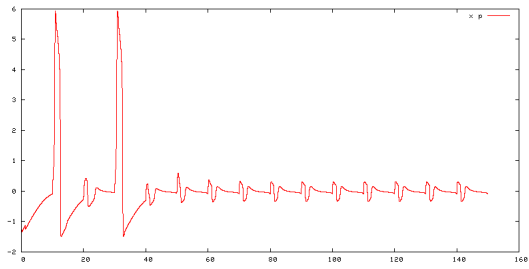


Figure 9:  $d = 11$ , big enough  $\lambda'$

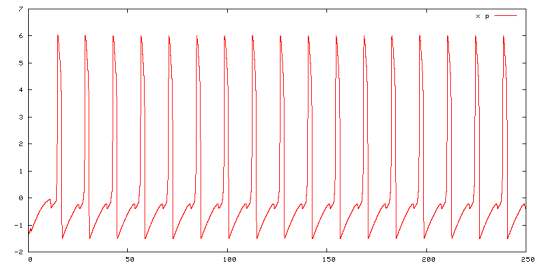
There is a main pyramidal neuron  $x_p$  connected to its inhibitory interneuron  $x_i$  with a synaptic delay  $d$ . We set an input  $u$  with a specific frequency. First we use a frequency close to the corresponding delay, see Figure 10b we see  $x_i$  which adapt to the input and then since the delay is of a close value,  $x_p$  stops to spike. In Figure 10d we kept the delay of Figure 10b but set a further input frequency.

The system :

$$\begin{aligned}\dot{x}_p &= f(x_p) + e + \lambda(u(t) - x_p) - w_i x_i(t - d) \\ \dot{x}_i &= f(x_i) + e + \lambda'(u(t) - x_i)\end{aligned}$$



(b)  $d = 11, \lambda' = 1.3$ , input period 10



(d)  $d = 11, \lambda' = 1.3$ , input period 14

## 7.2 Input continuity precisions

### 7.2.1 Lego game

The input continuity as introduced in Section 4 and defined at (5) allow us to play the Lego game : plug in serial and parallel blocks having input continuity and get a bigger block with input continuity. To plug in serial, the property of the output of the first block should of course imply the property needed by the input of the second block. The parallel block is just a redefinition of the input and input space using for example as a new distance the sup of the two original distances. Feedbacks are of a different kind of plug and we will need some more refined analysis for example in two steps : block A has two input, one is a feedback, if we can prove input continuity of A depending only on the first input, we have some property of the output, then knowing that, we have some property of the feedback input and we have a new ( and eventually stronger ) input continuity of A using the full property that we now know on the input, giving the full. This kind of computation is close to the classical idea of predictive top-down signal which is used to improve the treatment of the feedforward input signal.

### 7.2.2 Norms, distances and properties

There is a lot of different coding we can think of being used , like phase, frequency, timing, spatial, etc ( some interesting examples among thousands [37, 32] ) Each defining different "distances" between signals and natural properties we could be tempted to prove on signals. In general we will have a pseudo-distance in the signal space but a real distance in the property space, with the equivalence classes  $x \in [a] \iff d(x, a) = 0$ , representing signals with the same property. To generalize this idea, we can define a measure  $\varphi^t : \mathbb{R}^n \mapsto \mathbb{R}^m$  of our signal which describe some properties that is if  $\varphi^t(x_1) = \varphi^t(x_2)$  then  $x_1$  and  $x_2$  have the same property at time t, with this we have a generic pseudo-distance defined with the usual norm in the property space :

$$d_{\varphi^t}^t(x_1, x_2) = \|\varphi^t(x_1) - \varphi^t(x_2)\|_m^t$$

. This construction was used for the input continuity of systems contracting toward a linear subspace Section 4.2.2

*Remark 7.1.* The interest of using a norm over using a distance : when is available a norm coming from a dot product, we can define orthogonal projection on subspaces, giving minimal distance between a real input and the space of desired input.

### 7.2.3 Classical codes and distances

We can define several distances to represent classical coding or properties of neuronal networks :

- a pseudo-distance for frequency coding :

Considering the signals defined by their spiking times :  $t_{x_1}^k$  and  $t_{x_2}^k$ , let  $T_{x_1}^t$  be the mean distance between  $t_{x_1}^k$  and  $t_{x_1}^{k+1}$  before time  $t$  ( representing the mean period of spiking ) then we could use :

$$d^t(x_1, x_2) = |T_{x_1}^t - T_{x_2}^t|$$

- a pseudo-distance for synchrony in a certain subspace defined by a projector  $V$ , remark that this is the one we mainly use throughout the paper:

$$d^t(x_1, x_2) = |V(x_1) - V(x_2)|$$

Depending on the signal representation, we will have different classical distances :

- With signals as a set of spikes, described by the set of spike's dates : to signal  $u$  we associate  $\{\tau^u\}$  so that  $u(t) = 1$  if  $t \in \{\tau^u\}$  else 0. We can then define :

$$d^t(u, v) = \sum_{\tau_k \leq t \text{ in } \{\tau^u\} \cup \{\tau^v\}} (|u(\tau_k) - v(\tau_k)|) \text{ with } \tau_k \in \{\tau^u\} \cup \{\tau^v\}$$

This one gives us a very precise distance between discrete spatiotemporal patterns but is too sensitive.

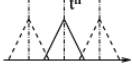
*Remark 7.2.* Something important to notice is the fact that the distances is defined at a time  $t$ , and may have access to the history of the signal. The property can be varying with the time, since the distance is.

- We can define sensitivity delay  $t_\epsilon$  also seen as refractory period, and use a kernel [38] :

$$u(t) = \sum_{\tau_k^u \text{ in } \{\tau^u\}} K(t, \tau_k^u, \{\tau^u\})$$

- $K(\cdot) = \delta(t - \tau_k^u)$  which gives a formalization of the above vision,
- $K(\cdot) = \delta(t - t_i^n)$  The spike train itself  $\sum_{t_i^n \in \mathcal{F}_i} \delta(t - t_i^n)$
- $K(\cdot) = W_{t_i^{n-1}..t_i^n}(t) \frac{t_\epsilon}{t_i^n - t_i^{n-1}}$  The normalized instantaneous frequency ( $t_i^n$  being predicted after  $t_i^{n-1}$ )

- $K(.) = \max\left(0, \frac{t_\epsilon - |t - t_i^n|}{t_\epsilon}\right)$  A non-causal measure of the instantaneous spike density.



- $K(.) = (1 - e^{-\frac{t_\epsilon}{\tau}})Y(t - t_i^n) e^{-\frac{(t - t_i^n)}{\tau}}$  A  $\tau$  time-constant, low-pass filtered causal measure of.
- $K(.) =$  if  $i = 0$  then  $\frac{t_\epsilon}{t_i^0 - t_\bullet}$  else 0 A representation of the 1st spike as in fast-brain mechanisms [39] with respect to a time reference  $t_\bullet$ .

and then we can define instantaneous distances with

$$d^t(u, v) = |u(t) - v(t)|$$

or define a more interesting distance as seen in the following, since thanks to the kernels we come back to real signal space.

- With the vision of spike set, we can consider the signals as binary words (1 if the time is a spiking time, 0 otherwise), and use the usual binary infinite word distances. This kind of norms can be useful if we are looking at some binary coded properties, or to do some binary computation [40]. Binary codes or barcodes [41].
- Some more statistical distances could be of interest, taking in account the probability of spiking with respect to the history by some Hebbian rule, for this we can be inspired by [42, 5, 43]
- Phase synchrony measure giving also a pseudo distance inspired by [19]

Distances in real signal space :  $\mathbf{R}^{\mathbf{R}}$ , we can come back to the usual functional norms, with  $T$  the sensitivity window :

$$N_2^t(u) = \int_{t-T}^t u(\tau)^2 d\tau \quad \text{or} \quad N_1^t(u) = \int_{t-T}^t |u(\tau)| d\tau$$

or the convoluted of these ones, with  $\mu \geq 0$

$$N_2^t(u) = \int_{t-T}^t u(\tau)^2 \mu(\tau - t) d\tau \quad \text{or} \quad N_1^t(u) = \int_{t-T}^t |u(\tau)| \mu(\tau - t) d\tau$$

And uses the norms as distances if we want with the usual :  $d^t(u, v) = N^t(u - v)$

We can list some properties of these norms :

- growing with  $T$  so the continuity with  $T$  implies the one with  $T' \geq T$
- $\mu$  will be very important to describe the system's sensitivity
- if we consider  $u$  to be bounded it is sufficient to have  $\mu \sim \frac{1}{\tau^2}$  to allow to have an infinite window  $T$

The norms are the generic case of the norm defined earlier Equation 9.

### 7.3 Contraction condition with help of symmetry

We have seen that with the contraction condition verified, we can control the symmetry of the dynamics to control the solution. Actually, the symmetry of the system can also help us to simplify the proof of contraction itself.

#### 7.3.1 Circulant functions, symmetries and contraction conditions

**Lemma 9.1.** *A  $\sigma$ -equivariance with  $\sigma$  a cycle of the size of the space is something well known:  $M$   $\sigma$ -equivariant iff  $M$  is circulant.*

*Proof.*  $M$  a  $m \times m$  matrix is circulant iff  $M$  seen as a quadratic form  $M(x, x) = x^T M x$  has the property  $M(x, \sigma x) = M(\sigma^{-1} x, x)$  with  $\sigma$  a  $m$ -cycle. On the other side we have  $\sigma^{-1} = \sigma^T$  and  $M(x, \sigma x) = x^T M \sigma x = x^T \sigma^T M x = (\sigma^{-1} x)^T M x$  by  $\sigma$ -equivariance.  $\square$

*Remark 7.3.* A matrix  $C$  is circulant iff it can be written as :

$$C = \begin{pmatrix} c(0) & c(1) & \dots & c(n-1) \\ c(n-1) & c(0) & c(1) & \dots \\ \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

The eigenvalue of the symmetric part are also simple :

$$\mu_i = \sum_{k=0}^{n-1} c(k) \cos(2ik\pi/n)$$

We will consider a  $\sigma$ -equivariant (circulant cf 9.1) system  $f'$  in space  $E'$  of size  $n'$ . We define

$$\lambda'_j = \sum_{k=0}^{n'-1} \partial_k f'_0 \cos(2jk\pi/n')$$

*Remark 7.4.* The  $\lambda'$  are defined using the first composant of  $f'$  but thanks to the circulant property, it could be equivalently done using any other composant

Then we can prove two interesting lemmas :

**Lemma 9.2.**

$$f' \text{ contracting with identity metric} \iff \forall j \in [0..n' - 1], \lambda'_j < 0 \quad (13)$$

**Lemma 9.3.**

$$V'_\sigma \frac{\partial f'}{\partial x} V'^{T}_\sigma < 0 \iff \forall j \in [1..n' - 1], \lambda'_j < 0 \quad (14)$$

*Remark 7.5.* We have a condition of size  $n' - 1$  as it should be.

*Remark 7.6.*  $V'_{all} = V'_\sigma$

With this two lemmas we have moreover a simple link between the contraction and the contraction toward the subspace of synchrony.

### 7.3.2 Discrete symmetric system

We consider  $f$  a  $\gamma$ -equivariant system :

$$f\gamma = \gamma f \quad (15)$$

We will consider that  $\gamma$  is a permutation and use notation from Section 3.1.  $l_i$  is the size of  $\sigma_i$ ,  $E_{I_q}$  correspond to the space invariant by the action of  $\gamma$  which is of dimension  $q = n - \sum_i l_i$  since the cycles  $\sigma_i$  are chosen non-trivial.

Such a  $\gamma$ -equivariant system will converge exponentially to a solution  $x = \gamma x$  if it has one of the contraction property. We will look at the strongest one : contraction toward  $\mathcal{M}_\gamma = \{x = \gamma x\}$ , and precisely at the sufficient condition  $V_\gamma \frac{\partial f}{\partial x} V_\gamma^T < 0$  with  $V_\gamma x = 0 \iff x \in \mathcal{M}_\gamma$ .

**Theorem 10.**

$$V_\gamma \frac{\partial f}{\partial x} V_\gamma^T < 0 \iff \forall i \in [0..p] \forall j \in [1 \dots l_i - 1], \lambda_{j,i} < 0$$

with

$$\lambda_{j,i} = \sum_{k=0}^{l_i-1} \partial_{k_i} f_{0_i} \cos(2jk\pi/l_i)$$



$k_i$  being the indice of the  $k$ th element in the  $i$ th subspace (modulo  $l_i$  to stay in the  $i$ th subspace). For example for  $i = 1$  we have  $k_1 = l_1 + k$ ,  $(-2)_1 = l_1 + l_2 - 2$  and  $f_{0_i} = f_{l_1}$

*Proof.*

*Remark 7.7.*  $x = \gamma x$  is equivalent to the conjunction  $\forall i \in [0..p] \ x|_{E_{\sigma_i}} = \sigma_i(x|_{E_{\sigma_i}})$ . But  $x|_{E_{\sigma_i}} = \sigma_i(x|_{E_{\sigma_i}})$  means that  $x$  is in synchrony inside each  $E_{\sigma_i}$  : We want to prove a polysynchrony which can be done by proving the synchrony inside each group.

We can relate this remark to

$$V_\gamma \frac{\partial f}{\partial x} V_\gamma^T < 0 \iff \forall i \ V_{\sigma_i} \frac{\partial f}{\partial x} \Big|_{E_{\sigma_i}} V_{\sigma_i}^T < 0$$

Which leads us to us the result with Lemma 9.3. □

## 7.4 Proofs of Theorems, Lemma etc

### 7.4.1 Theorem 7

*Proof.* We have the hypothesis:

$$\begin{aligned} \forall \tau \geq t_0, \ \|u(\tau)\| \leq M', \ \|s(\tau)\| \leq M' \\ N_{\alpha, \|\cdot\|}^{t_0, t}(u' - u) \leq \eta' \end{aligned}$$

Using the uniform continuity,

$$\begin{aligned} \forall \tau \geq t_0, \ \|h(\tau)\| \leq M \\ N_{\alpha, \|\cdot\|}^{-\infty, t}(h) \leq \eta \end{aligned} \tag{16}$$

We have

$$N_{\alpha, \|\cdot\|}^{-\infty, t}(r) = \int_{-\infty}^t e^{\alpha(\tau-t)} \|r(\tau)\| d\tau$$

However from the robustness (7)

$$\forall \tau \ \|r(\tau)\| \leq \int_{-\infty}^{\tau} e^{\lambda(y-\tau)} \|h(y)\| dy$$

We can apply Lemma 10.1 with  $t_0 = -\infty$  and define  $t_1$  the moment of the saturation

$$e^{\lambda(t_1-t)} = \frac{\eta\lambda}{M} \tag{17}$$

$$N_{\alpha, \|\cdot\|}^{-\infty, t}(r) \leq \lim_{t_0 \rightarrow -\infty} \int_{t_0}^{t_1} e^{\alpha(\tau-t)} \frac{M}{\alpha} (1 - e^{\lambda(t_0-\tau)}) d\tau + \int_{t_1}^t e^{\alpha(\tau-t)} \eta e^{\lambda(t-\tau)} d\tau$$

which by calculus using the relation Equation 17

$$\leq \frac{M}{\lambda\alpha} \left( \frac{\eta\lambda}{M} \right)^{\frac{\alpha}{\lambda}} + \frac{\eta}{\alpha - \lambda} \left( 1 - \left( \frac{\eta\lambda}{M} \right)^{\frac{\alpha-\lambda}{\lambda}} \right) \quad (18)$$

From (18) we have the continuity, since all the powers of  $\eta$  are positives.  $\square$

Let's consider some cases :

- if  $\alpha \leq \lambda$  We first have to say that we still have a positive term, since the exponent change also its sign, making  $\left( 1 - \left( \frac{\eta\lambda}{M} \right)^{\frac{\alpha-\lambda}{\lambda}} \right) \leq 0$  then the fact that  $M$  is finite is important, otherwise this term will go to  $+\infty$  and we can have in extreme cases some numerical surprises (even if we have continuity).
- if  $\alpha = \lambda$  then by continuity we get the limit :  $\frac{\eta}{\alpha} + \frac{\eta}{\alpha} \ln \left( \frac{M}{\eta\alpha} \right)$  so  $M$  should still be bounded to allow us to use this limit..
- if  $\alpha \geq \lambda$  then we can get rid of  $M$  and perhaps in first approximation, just keep the main term :  $\frac{\eta}{\alpha-\lambda}$

*Remark 7.8.* If  $t_1$  doesn't exist ( i.e.  $\frac{\eta\lambda}{M} \geq 1$  ) it first means that we did not took  $\eta$  small but the calculus gives just the first term of (18) which is still good.

*Remark 7.9.* We can also instead of using some case based calculus over the  $\lambda$  and  $\alpha$  use the lemma 10.2 to have a pseudo equivalence of all of these norms.

*Remark 7.10.* We should see that in the preceding proof,  $\lambda$  is taken as the contraction rate of the system, but we can use any  $\lambda' \leq \lambda$  since (6) will still be true.

**Lemma 10.1.** *a boundary on the norm before  $t$ :*

$$\forall t \geq \tau \geq t_0, \int_{t_0}^{\tau} e^{\beta(y-\tau)} H(y) dy \leq \min \left( \eta e^{\beta(t-\tau)}, \frac{M}{\beta} (1 - e^{\beta(t_0-\tau)}) \right) \quad (19)$$

*Remark :* The inequality is an equality for

$$H(y) = \begin{cases} M & \text{if } y \leq t_1, \\ 0 & \text{if } t_1 \leq y \leq t \end{cases}$$

with  $t_1$  the moment of saturation if it exists :

$$\frac{M}{\beta}(e^{\beta(t_1-t)} - e^{\beta(t_0-t)}) = \eta$$

*Proof.*

$$\begin{aligned} \forall \tau \geq t_0, \int_{t_0}^{\tau} e^{\alpha(y-\tau)} H(y) dy &= e^{\alpha(t-\tau)} \int_{t_0}^{\tau} e^{\alpha(y-t)} H(y) dy \\ &\stackrel{\text{def}}{=} e^{\alpha(t-\tau)} h(\tau) \end{aligned}$$

however  $h(t_0) = 0$ ,  $h(t) = \eta$ ,  $\dot{h}(\tau) = e^{\alpha(\tau-t)} H(y)$

$$\begin{aligned} \text{from (16)} \quad 0 &\leq \dot{h}(\tau) \leq M e^{\alpha(\tau-t)} \\ 0 &\leq h(\tau) \leq \min \left( \eta, \int_{t_0}^{\tau} e^{\alpha(y-t)} M dy \right) \\ 0 &\leq h(\tau) \leq \min \left( \eta, \frac{M e^{-\alpha t}}{\alpha} (e^{\alpha \tau} - e^{\alpha t_0}) \right) \end{aligned}$$

□

**Lemma 10.2.** *The possibility to change the  $\alpha$  of the norm keeping a boundary:*

$$\begin{aligned} N_{\beta, \|\cdot\|}^{-\infty, t}(d) &\leq q(\eta) \\ \text{with} \quad q(\eta) &\leq \eta \quad \text{if } \alpha \leq \beta \\ \text{and} \quad q(\eta) &\leq \frac{M}{\beta} \left( \frac{\eta \alpha}{M} \right)^{\frac{\beta}{\alpha}} \quad \text{if } \beta \leq \alpha \end{aligned} \tag{20}$$

*Remark :* it is also true that we keep the boundary with  $t_0 \neq -\infty$  but the result is less interesting.

*Proof.* With the notation of lemma 10.1 :

$$\begin{aligned} N_{\beta, \|\cdot\|}^{-\infty, t}(d) &= \int_{t_0}^t e^{\beta(\tau-t)} D(\tau) d\tau \\ &= e^{(\alpha-\beta)t} \int_{t_0}^t \dot{h}(\tau) e^{(\beta-\alpha)\tau} d\tau \end{aligned}$$

integrating by parts and taking the limit  $t_0 = -\infty$  when possible :

$$= h(t) + (\alpha - \beta) \int_{t_0}^t h(\tau) e^{(\beta-\alpha)(\tau-t)} d\tau$$

for  $\alpha \geq \lambda$  and using the property of  $h(\tau)$  seen in lemma 10.1

$$\leq \eta + (\alpha - \beta) \left[ \int_{t_0}^{t_1} \frac{M e^{-\alpha t}}{\alpha} (e^{\alpha \tau} - e^{\alpha t_0}) e^{(\beta - \alpha)(\tau - t)} + \int_{t_1}^t \eta e^{(\beta - \alpha)(\tau - t)} \right]$$

which gives us, with  $t_0 = -\infty$  ( which works without any new hypothesis)

$$\leq \eta + (\alpha - \beta) \left[ \frac{M e^{\beta(t_1 - t)}}{\alpha \beta} + \frac{\eta}{\beta - \alpha} (1 - e^{(\beta - \alpha)(t_1 - t)}) \right]$$

and using the relation (17)

$$\leq \frac{M}{\beta} \left( \frac{\eta \alpha}{M} \right)^{\frac{\beta}{\alpha}}$$

□

## 7.5 Proof of Lemma 9.3

*Proof.* A natural projector to the subspace of synchrony is  $W$ :

$$W = I - \sigma = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & -1 \\ -1 & 0 & \dots & 0 & 1 \end{pmatrix}$$

To use our natural projector, we first have to remark that it is not really a projector to the orthogonal space. Such a projector can be obtained by removing the redundancy inside  $W$ , because it is a system of dimension  $n$  but represent an hyperspace of dimension  $n - 1$ .

Then this noticed, instead of looking for the condition  $V'_{all} \frac{\partial f}{\partial x} V'^T_{all} < 0$  we will look for the equivalent condition  $W \frac{\partial f}{\partial x} W^T$  strictly negative except for one null eigenvalue.

$$\begin{aligned} A &= W \frac{\partial f'}{\partial x} W \\ &= (\partial_s f'_r - \partial_{s+1} f'_r - (\partial_s f'_{r+1} - \partial_{s+1} f'_{r+1}))_{(r,s) \in [0..n'-1]^2} \\ &= (2\partial_s f'_r - \partial_{s-1} f'_r - \partial_{s+1} f'_r)_{(r,s) \in [0..n'-1]^2} \text{ using that } \partial f' \text{ is circulant} \end{aligned}$$

*Remark 7.11.* All indices are modulo  $n'$ .

*Remark 7.12.*  $W$  is circulant.

Since  $A$  is circulant (product of circulant matrix), it is defined by

$$a(k) = A_{0,k} = 2\partial_k f'_0 - \partial_{k-1} f'_0 - \partial_{k+1} f'_0$$

We are interested in  $C$  its symmetric part. It is also a circulant matrix, so defined by  $c(k)$   $k \in [0..l_i - 1]$  :

$$c(k) = \frac{a(k) + a(-k)}{2} = \frac{1}{2} (2\partial_k f'_0 - \partial_{k-1} f'_0 - \partial_{k+1} f'_0 + 2\partial_{-k} f'_0 - \partial_{-(k-1)} f'_0 - \partial_{-(k+1)} f'_0)$$

One of the interest of circulant matrix is that we know their eigenvectors and associated eigenvalues : taking  $w'_j = e^{i\frac{2\pi}{n'}}$  one of the  $n'$ th root of 1 ( $w_j^{n'} = 1$ ), we construct the eigenvector

$$v_j = (1, w'_j, w_j'^2, \dots, w_j'^{n'-1}) \text{ associated to the eigenvalue } \mu'_j = \sum_{k=0}^{n'-1} c(k) w_j'^k$$

*Remark 7.13.* Since we took  $C$  symmetric ( $c(k) = c(-k)$ ), the eigenvalues  $\mu'_j$  are all real as one could remark regrouping  $c(k)w_j'^k + c(-k)w_j'^{-k} = c(k)\cos(2jk\pi/n')$ .

With some simple regrouping :

$$\begin{aligned} \mu'_j &= \sum_{k=0}^{n'-1} c(k) w_j'^k = 2(1 - (w'_j + w_j'^{-1})/2) \lambda'_j = 2(1 - \cos(2j\pi/l_0)) \lambda'_j \\ \text{with } \lambda'_j &= \sum_{k=0}^{n'-1} \partial_k f_0 (w_j'^k + w_j'^{-k})/2 = \sum_{k=0}^{n'-1} \partial_k f_0 \cos(2jk\pi/n') \end{aligned}$$

And :

$$\begin{aligned} \mu'_0 &= 0 \\ \forall j \in [1..n' - 1], \quad \text{sign}(\lambda'_j) &= \text{sign}(\mu'_j) \end{aligned}$$

Since  $W$  is of dimension  $n'$ , but represent a  $n' - 1$  dimensional space, corresponding to  $\mu'_0$ . Then by virtue of Theorem 1 of [24] the contraction to the subspace of synchrony  $E'_{all}$  is ensured with  $\forall j \in [1..n' - 1]$ ,  $\mu'_j < 0$  or equivalently  $\forall j \in [1..n' - 1]$ ,  $\lambda'_j < 0$   $\square$

## 7.6 Contraction toward $\mathcal{M}_{all}$ using symmetry

### 7.6.1 Proof of Lemma 4.4

Let's recall the definition of a partial isometry : it is an isometry from the orthogonal of its kernel to its image.

*Proof.* We take  $VV^T = I_{E_{\mathcal{M}^\perp}}$ ,  $V_2V_2^T = I_{E_{\mathcal{M}_2^\perp}}$ , and  $\forall x \in E_{\mathcal{M}^\perp}$ ,  $x^T \Theta V f V^T \Theta^{-1} x < 0$ .

Note first that  $V_2V^T \Theta^T \Theta V V_2^T > \mathbf{0}$ . Indeed,  $\Theta$  is invertible and  $\forall y \in E_{\mathcal{M}_2^\perp}$ ,  $V V_2^T y = 0$  implies  $V_2^T y = 0$  (since  $\mathcal{M}_2^\perp \subset \mathcal{M}^\perp$ ), which in turn implies  $y = 0$  (by definition).

Next note that

$$\exists \Theta_2 \in GL(E_{\mathcal{M}_2^\perp}), \forall y \in E_{\mathcal{M}_2}, \exists x \in E_{\mathcal{M}}, V^T \Theta^{-1} x = V_2^T \Theta_2^{-1} y \text{ and } V^T \Theta^T x = V_2^T \Theta_2^T y$$

Indeed, since  $VV^T = I_{E_{\mathcal{M}^\perp}}$  this is equivalent to :

$$\begin{aligned} \exists \Theta_2 \in GL_m, \forall y \in E_{\mathcal{M}_2^\perp}, & \quad \Theta V V_2^T \Theta_2^{-1} y = (\Theta^T)^{-1} V V_2^T \Theta_2^T y \\ \exists \Theta_2 \in GL_m, & \quad V_2 V^T \Theta^T \Theta V V_2^T = \Theta_2^T \Theta_2 \end{aligned}$$

where  $m$  is the dimension of the subspace  $E_{\mathcal{M}_2^\perp}$ ,  $\Theta_2$  exists because  $V_2 V^T \Theta^T \Theta V V_2^T > \mathbf{0}$ .

Finally, by unitary freedom of square roots for symmetric positive operators there exists a partial isometry  $U$  such that  $\Theta_2 = U \Theta V V_2^T$ . Thus,

$$\Theta_2 V_2 f V_2^T \Theta_2^{-1} = U \Theta V V_2^T V_2 f V_2^T (U \Theta V V_2^T)^{-1} = U \Theta V f V^T \Theta^{-1} U^T < 0$$

where the last expression is negative definite from the hypothesis.

As is  $\Theta_2$ ,  $U$  is defined up to isometries. We can thus take any  $U$  having the sufficient following properties.  $U^T$  is basically the partial isometry embedding  $E_{\mathcal{M}_2}$  in  $\text{Im}(\Theta V V_2^T E_{\mathcal{M}_2})$  which is of the same dimension but in a bigger space :  $U^T U$  projects  $E$  onto  $\text{Im}(\Theta V V_2^T E_{\mathcal{M}_2})$  and  $U U^T = I_{E_{\mathcal{M}_2}}$ .  $\square$

### 7.6.2 Proof of Lemma 4.3

*Proof.* Using a similar construction as in proof 7.6.1, with  $m$  the dimension of the subspace  $E_{\mathcal{M}^\perp}$ , we need

$$\exists \Theta_2 \in GL_m, \quad \forall y \in E_{\mathcal{M}^\perp}, \Theta V V_2^T \Theta_2^{-1} y = (\Theta^T)^{-1} V V_2^T \Theta_2^T y$$

which is equivalent to :

$$\begin{aligned}\exists \Theta_2 \in GL_m, \quad (V_2^T)^{-1} V^T \Theta^T \Theta V V_2^T &= \Theta_2^T \Theta_2 \\ (T^T)^{-1} \Theta^T \Theta T^T &= \Theta_2^T \Theta_2\end{aligned}$$

things works with  $T$  unitary, but if not unitary there are few chances for i to work.  $\square$

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